Stat 201: Introduction to Statistics

Standard 17: Probability – Counting Techniques

Multiplication Rule of Counting

- If a task consists of a sequence of choices in which there are **p selections** for the first choice, **q selections** for the second choice, **r selections** for the third choice and so forth
- Then, together, the task can be done in (p*q*r*…) different ways

Multiplication Rule of Counting w/ Replacement

- A South Carolina license plate is three letters followed by three numbers. How many unique plates can they make in this format?
- 26 letters in the alphabet
- 10 single digit numbers
- 26*26*26*10*10*10 = 17,576,000 plates

Multiplication Rule of Counting w/o Replacement

- Let’s say to ease reading of plates South Carolina changes its license plate to be three letters followed by three numbers, but there can’t be any repeats of letters or digits. How many unique plates can they make in this format?
- 26 letters in the alphabet
- 10 single digit numbers
- 26*25*24*10*9*8 = 11,232,000 plates. Notice this is a lot less than before!

Multiplication Rule of Counting w/o Replacement

- 26*25*24*10*9*8 = 11,232,000 plates. Notice this is a lot less than before!
- Here, we see two instances where we multiply digits decreasing by one, in order. It isn't bad here because there are only three choices in each instance but what if license plates had more letters or digits?
  - To solve this we introduce **factorials**

Multiplication Rule of Counting w/o Replacement

- A **factorial** of a number $n \geq 0$ is defined as n! = n*(n-1)*(n-2)*…*3*2*1

- n! = n* (n-1)!

- 0! = 1
  - This is the trap door for the previous rule

Permutations

- A **permutation** is an ordered arrangement in which r objects are chosen from n distinct objects
  - $r \leq n$
  - No repetition

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

Permutations

- For example, suppose we have a set of three letters: A, B, and C. We might ask how many ways we can arrange 2 letters from that set. Each possible arrangement would be an example of a permutation.
- The complete list of possible permutations would be: AB, AC, BA, BC, CA, and CB.
  - Here AB and BA are distinctly different!

Combinations

- A **combination** is collection, without regard to order, in which r objects are chosen from n distinct objects
  - $r \le n$
  - No repetition

$$ {}_{n}C_{r} = \frac{n!}{r!\,(n-r)!} $$

Combinations

- For example, suppose we have a set of three letters: A, B, and C. We might ask how many ways we can select 2 letters from that set. Each possible selection would be an example of a combination.
- The complete list of possible selections would be: AB, AC, and BC.
  - Here AB and BA are treated the same and just represented by AB

Permutations vs Combinations

- The distinction between a **combination** and a **permutation** has to do with the sequence or order in which objects appear.
- A **combination** focuses on the selection of objects without regard to the order in which they are selected.
- A **permutation**, in contrast, focuses on the arrangement of objects with regard to the order in which they are arranged.

Combinations Example

- At Wendy’s you can order a hamburger with the following toppings: Cheese, bacon, mayo, ketchup, mustard, pickles, onion, lettuce, tomato.
- **Q:** How many different burgers can you order at Wendy’s?
- **A:** To answer this question we will use combinations because the order the toppings go on doesn‘t matter

Combinations Example

- For any hamburger we have nine possible toppings – here are the easy parts
- **Q:** How many hamburgers can you make with zero toppings?
- **A:** Just one – the burger itself
- **Q:** How many hamburgers can you make with nine toppings?
- **A:** Just one – the burger with all nine toppings

Combinations Example

- So far we've found two possible hamburgers – all toppings and no toppings.
- **Q:** How many hamburgers can you make with one topping?
- **A:** Here we can find nine different hamburgers – each one with one of the nine toppings

Combinations Example

- So far we’ve found 11 possible hamburgers – all toppings, no toppings and 9 one-topping-each
- **Q:** How many other hamburgers can you make?
- **A:** Here is were it gets a little complicated - the left over combinations aren’t as easy to think of.

Combinations Example

Number of Toppings	Number of Possible Burgers
0	1
1	9
2	nrow(combinations(9,2)) = 36
3	nrow(combinations(9,3)) = 84
4	nrow(combinations(9,4)) = 126
5	nrow(combinations(9,5)) = 126
6	nrow(combinations(9,6)) = 84
7	nrow(combinations(9,7)) = 36
8	nrow(combinations(9,8)) = 9
9	1
TOTAL	512

Combinations Example

- In each row of the previous table we found the number of possible hamburgers for that number of toppings
  - There are 84 different hamburgers with 2 toppings
- **In total we have 512 possible hamburgers**
  - **Note: we could have calculated this by considering each topping as a yes or a no (2 possibilities)**

$$2^{\#\,toppings} = 2^9 = 512$$

Permutations Example

Sheldon: Well, this sandwich is an unmitigated disaster. I asked for turkey and roast beef with lettuce and swiss on whole wheat.
Raj: What did they give you?
Sheldon: Turkey and roast beef with swiss and lettuce on whole wheat. It's the right ingredients, but in the wrong order. In a proper sandwich, the cheese is adjacent to the bread to create a moisture barrier against the lettuce. They might as well have dragged this thing through a car wash.

Permutations Example

- At Wendy’s you can order a hamburger with the following toppings: Cheese, bacon, mayo, ketchup, mustard, pickles, onion, lettuce, tomato.
- **Q:** If my friend is insane and order matters, how many possible hamburgers are there?
- **A:** To answer this question we will use permutations because the order the toppings go on does matter

Permutations Example

- For any hamburger we have nine possible toppings – here are the easy parts
- **Q:** How many hamburgers can you make with zero toppings?
- **A:** Just one – the burger itself
- **Q:** How many hamburgers can you make with one topping?
- **A:** Nine – the burger with one topping has no ordering

Combinations Example

- So far we've found 10 possible hamburgers – all toppings and 9 one-topping-each
- **Q:** How many other hamburgers can you make?
- **A:** Here is were it gets a little complicated - the left over permutations aren't as easy to think of.

Permutations Example

Number of Toppings	Number of Possible Burgers
0	1
1	9
2	nrow(permutations(9,2)) = 72
3	nrow(permutations(9,3)) = 504
4	nrow(permutations(9,4)) = 3,024
5	nrow(permutations(9,5)) = 15,120
6	nrow(permutations(9,6)) = 60,480
7	nrow(permutations(9,7)) = 181,440
8	nrow(permutations(9,8)) = 362,880
9	nrow(permutations(9,9)) = 362,880
TOTAL	986,410

Permutations Example

- In each row of the previous table we found the number of possible hamburgers for that number of toppings when order matters
  - There are 72 different hamburgers with 2 toppings
- **In total we have 986,410 possible hamburgers**
  - **Note: this is a lot more than the 512 combinations!**

Partitions Rule

- Suppose we want to partition a single set of N elements into k sets with n_1 items in set one, n_2 items in set two, …, n_k items in set k, such that $n_1 + n_2 + \cdots + n_k = N$
- The number of possible partitions is:

$$\frac{N!}{n_1!\,n_2!\ \ldots n_k!}$$

Partitions Rule : Example

- Suppose we have 20 students and we want to break them up into groups to solve four problems. The group sizes should be {2, 5, 5, 8} assigning less to the easier problems and more to the harder problems.
- The number of possible partitions is:

$$\frac{N!}{n_1!\,n_2!\;\ldots n_k!} = \frac{20!}{2!\,5!\,5!\,8!} = 2{,}095{,}133{,}040$$

Counting Summary

Type	Description	Formula
Combination	A selection of objects from a set when the order in which the objects are selected doesn't matter.	${}_nC_r = \frac{n!}{r!\,(n-r)!}$
Permutation w/replacement	A selection of objects from a set when the order in which the objects are selected matters and an object can be selected more than once.	n^r
Permutation w/o replacement	A selection of objects from a set when the order in which objects are selected matters and an object cannot be selected more than once.	${}_nP_r = \frac{n!}{(n-r)!}$
Permutation of non-distinct items	The number of ways n objects can be arranged when they are broken up by kind	$\frac{n!}{(n_1! * n_2! *,,,* n_k !)}$